Math Ed Res J (2014) 26:79–99
DOI 10.1007/s13394-013-0112-5

ORIGINAL ARTICLE

Unit fractions in the context of proportionality: supporting students' reasoning about the inverse order relationship

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Received: 25 February 2013 / Revised: 21 September 2013 / Accepted: 9 December 2013 /
Published online: 8 January 2014

Abstract We analyze a classroom design experiment, conducted in a fourth grade classroom, that served to explore an instructional path in which the introduction of unit fractions and supporting proportional reasoning coincide. Central to this path is the use of means of support in which the objects that unit fractions quantify are not characterized as equal-sized parts of a whole, but as entities that are always separate from a reference unit. We argue that such a path is crucial for helping students develop deep quantitative understandings of fractions, where fraction quantities are, from the very start, linked to the reciprocal and multiplicative relations that their use implies. We focus on the first part of the design experiment in which we helped the students make sense of a concept that is important for initial fraction learning and proportional reasoning, the inverse order relationship among unit fractions.

Keywords Design experiment · Fractions · Proportional reasoning · Primary education

Introduction

Proportionality and rational numbers are two tightly interrelated concepts. A mature understanding of the one requires a mature understanding of the other. For instance, a relatively sophisticated understanding of rational numbers is necessary to make sense of how the size of something small can be defined in a multiplicative relation to the size

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of something big (e.g., the area of Mexico is 4/15 the area of Australia). Correspondingly, proportionality is necessary to understand the idea of *equivalence class of rational numbers* (Lamon 2007).

In the context of proportionality, unit fractions play an important role in quantifying the reciprocal of basic multiplicative relations, where the scalar operator is a whole number. Take for example the following statement: “The distance from Mexico City to Queretaro is *fivefold* the distance from Guadalajara to Chapala.” Enunciating the reciprocal relation requires a unit fraction: “The distance from Guadalajara to Chapala is *one fifth* the distance from Mexico City to Queretaro.” However, in classrooms, the reciprocity in multiplicative relations and proportionality are typically introduced much later than unit fractions. Indeed, some consider understanding of fractions as a precursor for pupils' development of proportional reasoning (Norton 2006).

In this paper, we propose and explore an instructional path, in which the introduction of unit fractions and supporting proportional reasoning coincide. We argue that such a path is crucial for helping students develop deep quantitative understandings of fractions, where fraction quantities are, from the very start, linked to the reciprocal and multiplicative relations that their use implies (Thompson and Saldanha 2003).

We analyze findings from a classroom design experiment (Cobb et al. 1997; Gravemeijer and Cobb 2006; Stephan et al. 2003) that aimed at supporting fourth grade students' understanding of unit fractions as numbers that quantify *relative size* (Thompson and Saldanha 2003). We focus on the first 5 of the 13 sessions of the experiment, in which we helped the students make sense of *the inverse order relationship among unit fractions* (Tzur 2007; hereafter referred to as *the inverse order relationship*). Understanding this relationship involves making sense of why and how the size of the entities quantified by unit fractions decreases as the number in the denominator grows. This idea is integral to understanding the reciprocal of a basic multiplicative relation, in which the scalar operator is a whole number, and is thereby central to proportional reasoning (Thompson and Saldanha 2003). At the same time, it has proven difficult for students to grasp (Behr et al. 1984; Streefland 1991; Tzur 2007) and is one of those ideas where employing thoughtful means of support can make, we believe, much difference to students' learning.

In our work, we employ the design experiment methodology (Gravemeijer and Cobb 2006) to investigate how specific instructional tools (e.g., specific task scenarios) support the emergence of envisioned forms of student reasoning in a classroom, or how they fail to do so. To situate our work, we first discuss current instructional approaches to supporting students' early reasoning about fractions and proportions, which are overwhelmingly grounded in equipartitioning task scenarios (e.g., equally sharing a baguette, where each piece of the baguette represents—at the same time—a unit fraction and a part of the reference unit). In the discussion, we build on the phenomenological analysis of fractions of Freudenthal (1983) and explain why our proposed approach departs significantly from equipartitioning task scenarios. Instead, our approach involves instructional tasks in which the entities that fractions quantify are always *separate from the reference unit*. We specify the advantages of introducing fractions in this way, rather than as parts of wholes, to help students understand fractions as numbers that account for multiplicative relations.

The results of the design experiment indicate that the explored approach was effective in helping the participating students make sense of the inverse order relationship. In the individual interviews, conducted after day 13 of the experiment, it was noticeable that all the 14 participating students responded to the tasks in quantitatively sound ways and articulated reasonable justifications for their responses. In addition, two of the students seemed to be readily capable of reasoning in ways consistent with the inverse order relationship, even when presented with problems that did not allude to the contexts used during the experiment, and without teacher support. We conclude the paper by discussing the significance and implications of these findings for teaching unit fractions in ways that encourage early proportional reasoning.

Conceptual and theoretical framing

It is important to clarify that much of the research on fractions and proportional reasoning has been conducted assuming cognitive perspectives (Behr et al. 1992; Lamon 2007; Post et al. 1993; Steffe and Olive 2010; Tzur 1999), where the primary focus rests on understanding (and modeling) the processes of students' learning. For instructional design purposes, we found it useful to approach learning from a situated perspective (Kirshner and Whitson 1997; Cobb 2000b) and view it as changes in the forms of students' participation in classroom mathematical practices (Cobb 2003). Within the perspective we adopt, we always interpret learning with respect to and as being shaped by means of support, which therefore constitute the explicit focus of our research. It is for this reason that when we discuss relevant literature, we bring to the fore the instructional strategies and types of tasks that the researchers used as they strove to elicit specific forms of cognitive activity in learners.

Inverse order relationship

Students' difficulties in making sense of the inverse order relationship are partially due to the fact that the way in which this relationship behaves stands at odds with the quantitative understandings that students develop when learning natural numbers, where a bigger number always denotes a larger amount. Some authors have considered that those prior understandings interfere with students' learning of rational numbers (Post et al. 1993; Streefland 1991; Baroody 1991). Others have conjectured that "children's fractional knowing can emerge as accommodations in their natural number knowing" (Steffe and Olive 2010, p. vii). Whatever the case, the literature shows that supporting students to make sense of the inverse order relationship is an instructional goal that is both important and not easy to attain.

Among the studies that have addressed the inverse order relationship, the classroom teaching experiment conducted by Tzur (2007) is of particular importance, since it had this relationship as its main learning goal. In his review of the literature, Tzur recognized three cognitive activities that scholars have capitalized upon to bring forth fractional conceptions: splitting, distribution, and iteration. Significantly, from an instructional vintage point, the three activities require helping students to conceive an amount of an attribute as being determined by the number of parts in to which a whole (or reference unit) is equally divided.

The first of these cognitive activities, *splitting*, consists of mentally decomposing a unit into equal-sized parts. Instructional tasks aimed at engaging students in this activity

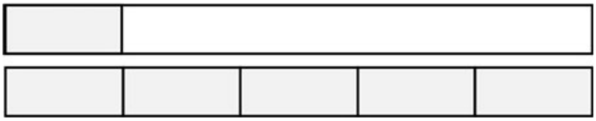

Fig. 1 *One fifth* portrayed as an equal-sized part of a whole, of such a size that when iterated five times it renders a length equal to the length of the whole

typically involve a narrative in which a food item (e.g., a blueberry pie) is divided into a certain number of equal pieces (Confrey and Maloney 2010). Reasoning about the notion of inverse order relationship is expected to emerge as students come to imagine what happens to the size of the parts, as the object is partitioned into increasingly more equal pieces (e.g., each fourth of a pie is further split into two equal parts).

The second cognitive activity, *distributing*, is closely related to the former one. It presents *fair sharing* as the main motivation for producing equal-sized rations. The main added feature is that it can involve acting on more than one item. Instruction aimed at engaging students in this activity typically entails a narrative in which one or several food items (e.g., hamburgers) are to be equally given out to a certain number of people (Streefland 1991; van Galen et al. 2008). Reasoning about the notion of inverse order relationship would be expected to emerge as students imagine what happens to the size of the parts, as the objects are fairly shared among increasingly more people.

The third cognitive activity, *iterating*, is the one in which the study of Tzur (2007) was based. The tasks designed to support this mental activity are different from the previous in two ways. First, the amounts to be partitioned are portrayed as lengths, irrespective of the task scenario, which could be about equally sharing a pizza. Second, iterating is used as a means of gauging the size of the equal pieces; both for making sure that the parts are equal in size and for comparing the size of a part with the size of the whole (see Fig. 1). In this approach, *iterating* (to make as much as a whole) and *partitioning* (a whole into equal size parts) are treated as inverse operations (Steffe and Olive 2010).

Several researchers have used the activity of iteration (Olive 1999; Steffe and Olive 2010; Tzur 1999). Their results show that instruction oriented to foster and take advantage of the iterating activity is a viable means of supporting students' fraction learning. However, these results also show that pupils that are supported in this way to learn fractions have to face several developmental hurdles that are rather difficult to clear (Norton and Hackenberg 2010).[1]

In his study, Tzur (2007) used the iterating activity to support students to make sense of the inverse order relationship by helping them conceive that as a whole is partitioned into more equal parts, more iterations of the length of one of the parts would be needed to create something as long as the whole. Tzur (2007) reported that he was fairly successful in promoting learning of the inverse order relationship, via the iteration

[1] It is worth clarifying that Norton and Hackenberg (2010) regard the need to face these hurdles not as being specific to a particular instructional strategy, but as a part of the process by which children develop increasingly sophisticated fraction conceptions. In the following section, we explain why, when assuming a phenomenological perspective, it is reasonable to regard the emergence of these hurdles as a function of using instructional approaches that capitalize on what Freudenthal (1983) called the "fraction as fracturer approach."

activity. After nine sessions, 12 of the 28 participating third grade students could readily reason in ways consistent with the inverse order relationship, and 16 more could reason about it when supported by a teacher. However, there were three pupils who had not formed the concept yet.

In the classroom design experiment we conducted, we aimed to help students to make sense of unit fractions—squarely and from the start—as numbers that account for proportional relations. In the following two sections, we explain why this oriented us to design all the means with which students' learning was to be supported within the *fraction as comparer approach* (Freudenthal 1983). Notably, this phenomenological approach to fractions involves supporting students' learning in a significantly different way than each of the three instructional strategies apparent in the studies just discussed. To outline the key differences, we first explain how the means of support used in the discussed strategies align with what Freudenthal (1983) termed the *fraction as fracturer approach*.

Fraction as fracturer

In his book *Didactical phenomenology of mathematical structures,* Freudenthal (1983) conducted an analysis in which he sought to identify *phenomena* that would "beg to be organized" (p. 32) by different mathematical concepts. This type of analysis is especially relevant to our approach to supporting student learning as it provides both an orientation and an initial rationale for designing specific task scenarios and other means of support that could be used in classrooms where the mathematizing of phenomena takes a central role.

In the case of fractions, Freudenthal identified two broad kinds of phenomena that would call for this concept, one of which encompasses the general model of a whole being divided into equal-sized parts: "In the most concrete way fractions present themselves if a whole has been or is being split, cut, sliced, broken, coloured in equal parts, or if it is experienced, imagined, thought as such" (p. 140). He coined the term f*raction as fracturer* to refer to instructional resources that are based on this general model of approaching fractions.

All the instructional strategies previously described, whether aimed at fostering the splitting, distributing, or iterating activities, would entail this phenomenological approach to fraction. As we explained above, these strategies all involve the use of task scenarios that call for associating unit fractions to the outcome of equally partitioning a food item (or some other kind of divisible object).

Freudenthal (1983) regarded the *fraction as fracturer approach* as being of "a convincing and fascinating concreteness" (p. 147). He also recognized in it some important phenomenological limitations that could inhibit multiplicative reasoning and fraction learning. In particular, he recognized that the action of equally partitioning a whole provides two aspects that are suitable for being mathematized: (1) the numerosity of the parts that are produced and (2) the size of each of the parts. From his phenomenological standpoint, he recognized the first of these aspects as the most concrete one. In contrast, he recognized the *magnitude aspect* as being less prominent, since it would mainly serve as a factor that checks "the fairness of the distributive procedure" (p. 149).

Freudenthal's analysis suggests that task scenarios based on the fraction as fracturer approach (e.g., food sharing scenarios) could easily prompt pupils to focus their attention on the physical outcomes of the actions that take place on the objects, rather than on what happens to the relevant magnitudes (e.g., the length of the pieces that are produced). This, in turn, could lead students to reason additively about the scenarios (e.g., reason about the numerosity of the discrete pieces into which a baguette was partitioned), and thus be inadequate for readily engaging them in reasoning about the relative size of specific entities.

Freudenthal also recognized in the fraction as fracturer approach "a quite restricted equivalence concept" (p. 147), since the number of identical things that are produced by dividing a whole into *n* equal parts is limited to *n*. As a consequence, this approach would not easily help students understand that the amount of an attribute quantified by a unit fraction could be iterated unrestrictedly. Instead, it would present a clear boundary for the number of iterations: the number of pieces into which the whole was divided. For this author, in phenomenological terms, fraction as fracturer "leads to proper fractions (<1) only" (p. 147).

Freudenthal's phenomenological analysis thus suggests that the fraction as fracturer approach would be inadequate for helping students make sense of fractions as numbers that can quantify the size of entities bigger than one. These kinds of fractions (i.e., improper fractions) are important in the context of proportionality since they are used in quantifying reciprocal relations of relative size, particularly when the scalar operator is not a whole number (e.g., if the cost of *product A* is 3/5 (i.e., three times 1/5) the cost of *product B*, then the cost of *product B* is 5/3 (i.e., five times 1/3) the cost of *product A*).

Empirical findings in fraction research are fairly in accord with Freudenthal's considerations. It has been documented that in developing increasingly sophisticated conceptions of fractions as parts of wholes, students have to clear several developmental hurdles (Norton and Hackenberg 2010). One of them involves coming to conceive a part of a whole as the size of an attribute, instead of as an actual piece of an object, or as a discrete element in a set. For instance, it involves coming to conceive 1/3 of a circle as an area of relative size to the area of entire circle, instead of as a physical piece of the represented circle, or as a subset of the set of discrete elements that is created when partitioning an object into three equal parts (Saxe et al. 2005). This first hurdle is consistent with Freudenthal's consideration about how discrete elements (often physical objects) can become the prominent issue to be mathematized, when a unit fraction is represented by a part that belonged to a partitioned whole (e.g., a slice of a baguette).

A second developmental hurdle involves conceiving the size of an attribute embodied by an equal-sized part as susceptible of being iterated unrestrictedly (i.e., beyond the whole). For instance, it involves coming to conceive 1/3 of a circle as the size of area that can be iterated unrestrictedly, instead of three times only (Tzur 1999). This second hurdle is consistent with Freudenthal's consideration about the fraction as fracturer approach leading to proper fractions only.

Fraction as comparer

Importantly, the equal partitioning of wholes is not the only kind of phenomena that asks to be mathematized by fractions. Freudenthal noticed that, in everyday situations, fractions "also serve in comparing objects which are separated from each other or are experienced, imagined, thought as such" (p. 145). For instance, fractions are used to express that a desk is 4/3 as long as the width of a window. Freudenthal coined the term *fraction as comparer* to refer to instructional resources that are based on this way of approaching fractions.

Task scenarios based on the fraction as comparer approach not only call for the use of fractions, they also place at the foreground *size*, rather than *numerosity*, as the attribute to be quantified (e.g., the size of certain lengths). In addition, they invoke the notions of multiplicative relationships (e.g., a desk measuring four of something of which the window is three), reciprocal relationships (e.g., the window being three times as wide as a fourth of the desk length), and allow for viewing unit fractions as being susceptible to being iterated unrestrictedly (specifically, beyond the measure of one whole). Based on Freudenthal's analysis, these instructional tasks thus seem to have the potential to guide learners towards sophisticated fraction notions, and, at the same time, encourage early focus on proportional reasoning. In the remainder of the paper, we describe and analyze the initial empirical testing of an instructional approach where *fraction as comparer* situations were used to support initial fraction learning and proportional reasoning.

A fraction as comparer approach to unit fractions

In describing our design experiment, it is important to note that we found no instructional strategy that would use *fraction as comparer* as a systematic approach to supporting initial fraction learning or proportional reasoning. Our design goal was to propose and test instructional activities and tools that would be consistent with this approach. The specific pathway we explored involves supporting students, early on, to imagine unit fractions as numbers that account for the size of a specific attribute (e.g., length) of a thing, which is separate from a reference unit. To this end, we used a wooden stick (about 24 cm long) to represent the reference unit and a set of colored plastic drinking straws that could be cut into specific unit fraction lengths (see Fig. 2). The specific way to imagine unit fractions we aimed to cultivate involves a set of given lengths that each fulfill an iterative condition with respect to a reference unit.

A straw would be one half as long as the stick when being of such a size that two iterations of its length would be necessary to cover the exact length of the stick (see Fig. 3). A straw would be one third as long as the stick when being of such a size that three iterations of its length would be necessary to cover the exact length of the stick,

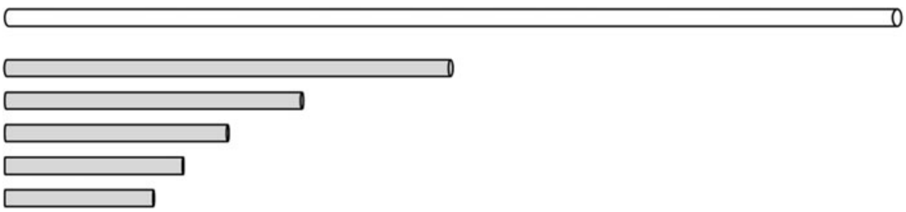

Fig. 2 The fractions 1/2, 1/3, 1/4, 1/5, and 1/6 as the sizes of rods independent of the reference unit

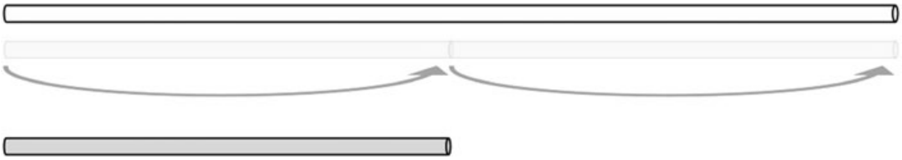

Fig. 3 A rod of such a size that two iterations of its length render the length of a stick

and so on. The reader will notice that a unit fraction here stands for the specific size of something that is *not* a part of the unit whole.

To produce the straws of desired lengths, their initial lengths would have to be reduced or enlarged, iteratively, so as to fulfill a specified condition. For instance, a student could cut a straw to the estimated length of *one half*. Let us imagine that, when iterating the length of this straw two times, the resulting length surpassed the length of the stick. The student would then need to decide how to adjust the length of the straw (cut it shorter or take a somewhat longer straw instead) so that exactly two iterations would match the length of the stick. Through a process of repeated trials, the specified size would be reached.[2]

The reader will notice some similarities between the instructional approach we propose and the one derived from the iterating activity, used by Tzur (2007). Indeed, both approaches make use of *length* as the attribute on which to focus students' quantitative reasoning. In addition, in both, the iteration of the lengths of entities that are separate from the reference unit is expected to serve as an important means of supporting students' reasoning about the relative size of unit fractions. However, Tzur used task scenarios consistent with the fraction as fracturer approach, in which pupils were asked to imagine a situation in which a "French Fry" (represented by a paper strip) was to be equally shared by a certain number of children. The entities that unit fractions would quantify (the size of a piece of a French Fry) were a part of a containing whole. Iteration was introduced as a means of helping pupils to reason about the absolute and relative size of these parts.

In contrast, in the approach we propose, the expectation is that unit fractions will be regarded as numbers that account for the size of entities (i.e., the length of plastic straws) that are always seen as being separate from the reference unit. The number in the denominator is not expected to denote the number of equal size parts into which a whole is partitioned. Instead, it is expected to allude directly to the number of iterations that an entity would require to produce something as big as one.

Elsewhere, we have broadly discussed the instructional opportunities that the comparer approach could offer for supporting students' development of increasingly sophisticated fraction notions (Cortina et al. 2012). Regarding the specific case of the inverse order relationship, the comparer approach has not, to our knowledge, been previously used to support students in making sense of this concept. Given that it is an important concept with regard to initial fraction learning and proportional reasoning, as well as one difficult to grasp, we view

[2] In mathematical terms, this way of approximating the size of the straws so that they fulfill the specified iterative condition could involve an infinite process. However, in a physical world, the process ends when a good enough approximation is reached.

investigating the viability of the comparer approach in helping students make sense of this concept to be a worthwhile endeavor.

Data collection and data analysis

The study was conducted following the classroom design experiment methodology, as developed by Cobb and Gravemeijer (Gravemeijer and Cobb 2006; Stephan et al. 2003; Cobb 2000a). It was conducted in a single classroom and consisted of 13 instructional sessions, each lasting about 90 min. The main learning goal during the first five sessions consisted of helping students make sense of the inverse order relationship.

Fourteen students participated in the design experiment. Eight of them were girls and six boys. These children formed the only fourth grade classroom in an urban public school. All the sessions were videorecorded with two cameras. The first author of this paper was in charge of most of the teaching, and kept a set of field notes.

In consistency with the classroom design experiment methodology, the main purpose of the analysis was to create an account of the evolution of the *classroom mathematical practices* that emerged in the course of the instructional sessions, as well as of the diverse ways in which students participated in those practices (Cobb 2003). As we discuss below, reasoning about the size of subunits of measure, in ways consistent with the inverse order relationship, became the first mathematical practice to emerge. The analysis also involved accounting for the means by which the emergence of the classroom practices was supported.

Throughout the study, data were collected that allowed the research team to document the evolution of learning, both at the level of the whole classroom community and of the individual students. The former goal was accomplished by developing an account of the ways of reasoning, arguing, and symbolizing that became normative in the classroom community[3] (Cobb et al. 2001). This account was grounded on the videorecordings of all the sessions, as well as the set of field notes.

In addition, all the students' work was collected, and each of the participating pupils was individually interviewed twice. The first interview took place the day before the classroom design experiment started, and the second, the day after the thirteenth and last session. These interviews allowed the team to document not only the kinds of tasks that each student could and could not solve, but also the way in which each pupil construed the tasks and reasoned about them.

Students' prior instructional experiences

Instructional practices in the school in which the design experiment was conducted were fairly typical of institutions serving marginalized communities in Mexico. They entailed “substantial amounts of rote activity” (Ayon, quoted in Boaler 2002, p. 255),

[3] Following the design experiment methodology, ways of reasoning, arguing, and symbolizing were considered to have become normative when treated by the classroom community as being clear and beyond justification (Cobb et al. 2001)

and scarcely any of what professional associations of mathematics educators (e.g., Australian Association of Mathematics Teachers 2006; National Council of Teachers of Mathematics 2000) regard as *quality mathematics teaching*.

Students had received formal instruction in fractions since third grade. However, by the beginning of the design experiment—the middle of fourth grade—they seemed to have made little sense of this concept. In the individual interviews that were conducted one day prior to the first instructional session, a problem was included that involved a chocolate bar, represented by a rectangle of 5×10 cm. Students were shown cards with the inscriptions $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{2}{4}$, and were asked to identify on the bar the amount of chocolate that would correspond to each of them. In the case of 1/4 and 2/4, they were also asked to explain if it would be more, less, or the same as one half.

Eleven of the 14 students correctly recognized that the $\frac{1}{2}$ inscription would represent half of the bar. The remaining three were unsure about its meaning. The $\frac{1}{4}$ inscription was interpreted as number representing less than half of the chocolate bar by five of the students. Six other students thought it would be more, and the remaining three were not sure. Only one student recognized the $\frac{2}{4}$ inscription as representing half of the chocolate bar. As can be noticed, almost none of the students seemed to have yet developed adequate understandings about the quantitative meaning of common fraction inscriptions.

Hypothetical learning trajectory

In accord with the design experiment methodology, in the planning phase of the study, we formulated a hypothetical learning trajectory (Gravemeijer and Cobb 2006). In it, we specified the mathematical learning goals, envisioned the learning process by which they could be reached, and developed conjectures about possible means of support (Cobb 2003).

We chose to start with instructional activities that involved measuring length with nonstandard units, such as feet and hands (e.g., measuring the length of a rug using feet). We expected that these activities would be helpful in orienting students to reason in ways consistent with the inverse order relationship; for instance, to reason about the size of a foot relative to the number of iterations required to cover the length of a rug.

To help students construe the instructional activities as reasonable and worthy of their engagement, we planned to use a narrative about how an imaginary group of ancient Mayan people,[4] the Acahay, came to measure lengths with precision. Activities would follow in which students used parts of the body to measure.

The first issue we expected to problematize with the students was the need for defining a standardized unit of measure. We planned to engage pupils in whole-class conversations that focus on how and why measurements differ when different people measure the same thing using, for instance, their hand spans.

The following step would involve introducing the Tikhe, a wooden stick about 24 cm long, as the unit chosen by the *Acahay* to measure lengths. We conjectured that,

[4] The city in which the design experiment was conducted is in the region where Mayan peoples live.

at this point of the trajectory, students would come to consider the use of a standardized unit of measure as a solution to a problem they engaged with earlier.

The second main issue we expected to problematize with the students was the fact that a unit of measure like the Tikhe, by itself, was insufficient for accounting for the exact length of many things (e.g., the height of a chair), given that the length of those things did not correspond to the an exact number of iterations of the unit. We planned to achieve this objective by asking students to measure several things using the Tikhe and by bringing forward the issue of measuring with precision.

The activity of producing the straws (see above) would be introduced as the solution that the Acahay came up with for measuring lengths smaller than one Tikhe. Students would then be given straws and scissors, and asked to produce the *small of two* (i.e., of such a size that two iterations of its length would exactly cover the length of the Tikhe). The production of other smalls would follow: *small of three*, *of four*, *of five*, and so on (i.e., 1/3, 1/4, 1/5…). We conjectured that the production of the smalls would allow us to engage students in whole-class discussions, addressing issues related to the inverse order relationship. For instance, we could discuss what would be longer, a *small of twelve* (1/12) or a *small of fifteen* (1/15), without producing these smalls. It would also help us establish *the smalls* (unit fractions) as being of *specific length* relative to the Tikhe (a reference unit).

It is worth clarifying that some of the activities that we planned to use in this part of the trajectory have a close resemblance with activities that other researchers used. In particular, van Galen et al. (2008) describe an instructional sequence for teaching fractions that centers on the use of a measuring bar ("the Amsterdam foot"). However, in their sequence, subunits are portrayed as subdivisions of the bar, instead of as entities separated from the bar. Hence, students are oriented to construe unit fractions in ways that lie within the fraction as fracturer approach.

In the teaching experiments of Davydov (1969/1991), students were supported to develop fraction understandings as they reasoned about the measurement of lengths. Nonetheless, in this case also, unit fractions were mainly portrayed as equal sized parts of a unit whole. To our knowledge, instructional strategies consistent with the fraction as comparer approach, for helping students make sense of the inverse order relationship, have not been previously tested.

In a second phase of the trajectory, we anticipated that *the smalls* could become construed as fractional units of measure, in their own right. Equivalencies between the iteration of these subunits and the Tikhe (e.g., two iterations of the small of two would render a length equal to that of the stick) could serve as basis for comparing fractions. For instance, students could be asked to think about what would be longer, a strip whose length corresponded to four iterations of the small of three, or one that corresponded to five iterations of the small of six (i.e., 4/3 vs. 5/6).

Actual learning trajectory

Using parts of the body to measure

The first mathematical practice emerged between days 1 and 4, and consisted of reasoning about the relative size of unit fractions in ways consistent with the inverse order relationship, as we specified it earlier. The emergence of this practice involved an important shift in the way in which most of the students construed the relation between the size of a unit of measure and the number of iterations it required to cover a certain length. At the beginning of the instructional intervention, most of the pupils seemed to rely on what Behr et al. (1984) called *the whole-number-dominance strategy*. They regarded numbers that would come later in the counting sequence as always accounting for larger sizes.

The instructional activities used at the start of the experiment involved measuring lengths using parts of the body, such as feet and hands, and thinking about the size of the units that were used. On day 1, students were presented with two such problems. The first involved using their feet to measure the length of a rug. In the second, students were asked to use their hand spans to measure the length of a paper strip, pasted on the chalkboard. They were told that this strip represented the width of a window in a historical landmark building in the city.

Most of the students initially drew on the whole-number-dominance strategy when elaborating their responses (e.g., the more iterations of a hand span that were necessary to cover the length of the window, the longer the hand span). There were, however, some students who readily produced solutions to the problems that were consistent with the inverse order relationship; solutions that they could soundly justify. The public discussion of these solutions, as well as the opportunity afforded by the instructional activities to test empirically conjectures about both the size of a unit and the number of iterations it would require to cover a distance, played an important role in the emergence of the first mathematical practice.

It is worth clarifying that in these initial activities the whole lengths that served as a referent for assessing the relative size of the units of measure were not treated, themselves, as units of measure. Consequently, the conceptions developed by students as they reasoned about these problems were not yet fractional—nor were they intended to be so. Our research conjecture (one that we can now support with data) was that the initial conceptions students develop while reasoning about these problems would later become helpful to them when reasoning about the relative size of unit fractions. This reasoning would be consistent with the inverse order relationship: that the relationship between number of iterations needed to get as much as a reference unit and the size of the unit fraction that is iterated must, necessarily, be inverse.

The measurement of the paper strip that represented the length of a window served as a context for the first whole-class discussion in which issues related to the inverse order relationship were addressed. The students first used their own hand spans to measure the paper strip and recorded the obtained measures in their notebooks. Generally, they seemed to interpret the measures as accounting for an *accumulation of distance*.[5] Nonetheless, it became apparent that students did not consider it necessary to account for the space that a unit of measure would not cover exactly. Most of the children reported having measured a whole number of hand spans (e.g., seven). Others accounted for lengths

[5] Students interpreted the measures as signifying a series of nested distances, so that when a unit is iterated, "the distance covered by the first two units is nested in the distance covered by three units, and so on" (Stephan et al. 2003).

shorter than their hand span using somewhat vague expressions, such as "and another bit" ("y otro cachito"). As we elaborate below, this issue became the focus of instructional efforts later in the sequence.

After recording their measures of the paper strip, students were presented with a problem in which they were told that, when visiting the historical landmark building, two of the research assistants, Miss Claudia and Miss Renata, had measured the window using their hand spans. The measure obtained by Miss Claudia was of exactly five hand spans, and the one obtained by Miss Renata was of exactly six had spans. Students were asked to think about which of the two persons had the longer hand span. Only Miss Claudia was in the classroom at that time.

As the conversation unfolded, it was noticeable that at least half of the students relied on the whole-number-dominance strategy to interpret the situation. The following excerpt is representative of this way of reasoning:

> Teacher: Ah, look, some of you think that Miss Renata and others that Miss Claudia. Lourdes,[6] why do you think that Miss Renata has the bigger hand?
>
> Lourdes: Because she got six hand-spans ["seis cuartas"]?
>
> Teacher: Because she measured six? Because six is a bigger number than five? Is that why?
>
> Lourdes: The thing is that Miss Renata, her hand is bigger and that is why she gets six hand-spans.

The extract illustrates how some of the students directly associated the bigger unit to the bigger number. Hence, they regarded Miss Renata as the person having the longer hand span.

The teacher then asked a student who concluded that Miss Claudia had the bigger hand to explain her reasoning:

> Marisol: Because Miss Claudia has the bigger hand and she has to get fewer [hands as a result], because Miss Renata has the smaller hand and she gets more [hands].

Marisol's explanation was significant not only because it was consistent with the inverse order relationship, but also because it seemed to have a strong quantitative grounding. It is noticeable how she repeatedly referred to sizes: "bigger" (más grande), "gets fewer" (salirle menos), "smaller" (más chica), and "gets more" (le sale más). This explanation illustrates how the instructional activities that consisted of measuring lengths using parts of the body allowed some students to readily conceive the relation between the size of units of measure and the number of iterations they required to cover a given length in ways that were consistent with the inverse order relationship.

When the teacher asked the class about Marisol's reasoning, it became apparent that several of the students were experiencing difficulties in making

[6] All students are referred to in the paper using pseudonyms.

sense of it. The following conversation seemed to help most of the students to reason in ways consistent with the inverse order relationship in this session:

Teacher: So who do you think has the bigger hand, Miss Claudia or myself?

[Both adults show their hands to the students. It is clear that the classroom teacher has a bigger hand.]

Teacher: So let's see. If I measure it. A question. Let's see, let's think about this question. If I have a bigger hand than Miss Claudia, when I measure the strip will I get more or less than six, more or less than five hand-spans?

Some students: Less than five.

Teacher: Who thinks that more? [Nobody raises his/her hand]. Who thinks that less? [Several of the students raise their hands. Some do not.] Should we try?

Student: OK.

[The teacher measures and obtains three hand-spans and a bit more.]

Teacher: So let's see. What if somebody uses ten hand-spans to measure this [pointing at the paper strip]. Will he have a very big hand or a very small one?

Student A: Small.

Student B: Very tiny.

Student C: A baby.

Regarding the emergence of the first mathematical practice, this conversation was significant, as it was the first time that a solution consistent with the inverse order relationship was treated by the class as self-evident. It is noticeable that none of the students raised their hand when asked who thought that the bigger hand would fit more times in the paper strip. Likewise, no one expressed that the hand that would fit ten times, would have to be big. In contrast, several of the students appeared to have developed a rather clear image of the size of the hand spans. The comment of the baby hand, in particular, is consistent with the conjecture that many of the students were reasoning about the problems quantitatively.

It is possible that some of the children thought that the bigger hand would fit more times in the paper strip, but chose not to make it public. However, it is worth mentioning that the teacher worked to establish a classroom environment in which students felt free to make mistakes. Pupils were constantly encouraged to express non-understanding, and received positive feedback from the teacher when they did.

The conversation also illustrates an important aspect of the activities that involved measuring using parts of body. These activities allowed for testing—empirically—conjectures about the relation between the size of a unit of measure and the number of iterations it took to cover a length. These empirical tests seemed to help the students make sense of solutions that were inconsistent with the whole-number-dominance strategy.

Two more problems of the same type were presented to the students on the following day. Both times, solutions consistent with the inverse order relationship came to be treated in the classroom as self-evident. Generally, it seemed that most of the students had developed quantitative conceptions that made it possible for them to reason in ways consistent with the inverse order relationship.

Using a standardized unit of measure

As we explained above, the main activity we planned to use to engage students in fraction reasoning involved using a stick as a unit of measure and straws as subunits. In the latter part of day 1, we introduced a narrative about how ancient Mayan people measured, preceded by a rather lengthy conversation about life in southern Mexico, before the Spaniards arrived.

On day 2, the classroom discussed how humans first measured using parts of their bodies. The teacher then introduced the Tikhe as the unit of measure that could have been used by one of the ancient Mayan groups, the Acahay. A conversation was orchestrated in which the advantages of using a standardized unit of measure, over use of parts of the body, were discussed. Most of the children seemed to recognize that it was a good idea to have a standardized unit.

Each child was given a Tikhe and asked to use it to measure the length of four different things. When recording their measures, it became apparent that the students easily omitted the lengths that the Tikhe would not cover exactly, like they had done when measuring with their hand spans.

In terms of our instructional agenda, this was particularly troublesome, given that we had expected to introduce the production of fractional units as a means of accounting—exactly and systematically—for those lengths. We designed several activities intended to draw students' attention to those lengths, and to help pupils recognize the importance of accounting for them in a precise way. In the first activity, paper strips, whose length corresponded to each student's height ,were cut and given to the students who were asked to measure their strip with the stick.

As the students shared with the class the measures they obtained, it became noticeable that many of them had recorded “5 Tikhes”, although it was clear that some of the children were taller than others. Two students recorded their heights as “5 and another bit” but were also clearly not of the same height. The teacher made these inconsistencies noticeable to the class.

The second activity entailed reproducing a given length. Each of the students was given a paper strip of a different length. Once they measured it with the Tikhe and recorded the result, the original paper strip was set aside and the students were asked to cut a new strip of the same length as the first one. The lengths of the two paper strips were then directly compared.

To facilitate the measurement of the strips, students were given plastic straws that they could cut and use as an additional measuring aid. Most of the children cut a straw of a similar length to the one on the paper strip that the stick would not cover exactly. The students that used this strategy were asked to make a drawing in their notebooks of the exact length of the straw they had used.

The first time this activity was completed, several of the students ended up with strips of unequal sizes. However, the second time, almost all the students produced a strip of very similar size to the original one. Altogether, these activities allowed the teacher to focus students' attention on the lengths that the stick would not cover exactly. They also served to illustrate how such lengths could be accounted for using units of measure smaller than *one*.

Reasoning about the size of fractional units

On day 3, the teacher initiated a conversation about the possible drawbacks of creating a special subunit of measure for every length that the stick would not cover exactly. He tried to steer the class to recognize that, although this strategy made it possible to measure with precision, it was impractical, since it made it difficult to communicate the obtained measures: if each unit were to have its own name, there would be too many names to remember and it would be very difficult to systematically associate the names with the particular size of each unit.

The activity of producing the straws was then introduced as a way of creating subunits in a systematic way. Students were asked first to produce a *small of two*. They were told that this rod needed to be of such a size that when used to measure the Tikhe, the measure would have to be exactly two. To illustrate this, the teacher arbitrarily cut a small straw and iterated it on the stick. It was noticeable that after three iterations, the small straw would still not cover the length of the Tikhe.

The teacher asked the class if the straw could be a small of two. Only negative responses were uttered. Then the teacher asked if the small of two would have to be longer or shorter than the straw he used. All the students that responded said that it would have to be longer. When the teacher explicitly asked if anyone thought that it would have to be shorter, none of the students responded. Generally, students seemed to readily draw on the conceptions developed when reasoning about the relative size of the hand spans, and other body parts, when reasoning about the relative size of the straws.

All the students were then given a plastic straw, slightly shorter than the stick, and asked to produce a small of two. They were told that if necessary, they could use more straws. To complete the task, they needed to end up with a single straw and discard all leftovers.

It took about 15 min for all the students to make their small of two. During this time, the teacher and two of the research assistants helped the students to complete the task. They reminded the pupils that the straw they had to produce should be of such a length that when used to measure the Tikhe, the measure would have to be exactly two.

Once all the students had produced a small of two, the teacher asked the class if this subunit, together with the Tikhe, would suffice to measure with precision. Several of the students gave a negative response. Then, the teacher asked which other small might the Acahay have made. Vanesa responded: “The one of three?” The teacher said that that was exactly what the Acahay did. The following conversation then took place:

> Teacher: Which one would be bigger, the small of two or the small of three? Let's think.

Alejandro: I know.

Teacher: [Talking to Vanesa] Let's see if this is how you imagined the small of three. It would be a small that when we measure the stick with it, it would measure…

Vanesa [and other children]: Three

Teacher: Three. Would it be longer than the small of two or shorter?

Marisol: Shorter. [Other students also respond that it would be shorter]

Teacher: Who is not sure?

[Nobody responds]

Teacher: Who thinks it's smaller?

Vanesa: The one of three than the one of two?

Several students: The one of three, the one of three

Teacher: The one of three is smaller? What do you think Miguel Ángel?

Miguel: Three

Teacher: Let's see, let's listen why Miguel thinks that the one of three is smaller.

Miguel: [Gesturing three iterations on the Tikhe] Because it has to come out three, and it's tiny.

This excerpt is representative of how students reasoned about the relative size of the subunits, in ways consistent with the inverse order relationship. It is noticeable how, as a collective, the class accepted to regard the units associated with the bigger numbers as being smaller. In addition, several of the students seemed to anticipate the size of the new subunit, based on the number of iterations it would require to cover the length of the reference unit (i.e., the Tikhe). The case of Miguel is noticeable, as he was one of the students who initially drew much on the whole-number-dominance strategy. Now, he not only uttered a solution consistent with the inverse order relationship but also seemed to anticipate correctly the size of the subunit students were asked to produce next.

Producing the small of three took the rest of day 3 (about 17 min). The next day, students continued by producing the small of four. When the teacher asked the class if it would be longer or shorter than the small of three, only Ángel overtly responded that it would be longer. The teacher then inquired if there were other students that agreed with Ángel, and found none. When the teacher asked the class why the small of three would have to be longer than the small of four, the question was treated as obvious. For instance, Vicky said “Because it fits three times”.

The students then continued to produce more smalls. By now, several of them had become quite good at making them. The teacher tried to ensure that all the students produced at least four smalls. However, the children were allowed to make as many smalls as they wanted, as long as they continued with the sequence (i.e., small of five, of six, and so on).

Some of the students worked in pairs and others individually. As the teacher walked around the classroom, exchanges occurred that illustrate how several of the students were developing a rather comprehensive understanding of the inverse order relationship. Vicky and Lourdes' exchange took place when they already made a small of eight:

> Teacher: [Talking to Vicky and Lourdes] You are trying to get to 100! What size would a small of a 100 be?
>
> Lourdes: Tiny.
>
> Teacher: Tiny? But 100 is a big number.
>
> Viky: It [the small] needs to be tiny [gesturing with her fingers the size of something smaller than 1 cm] so that it [the Tikhe] fits 100 [of them].

Some of the students could now anticipate fairly well the size of a subunit specifying only the number of iterations it would require to cover the stick. This illustrates that each small became to have a specific length to these students, relative to the length of the stick. It is worth noticing that the actual production of the small mentioned by the teacher, with the tools at hand, would have been all but impossible. However, the students seemed capable of imagining the process of producing it, and of anticipating the size of subunit that would be created.

In the following days, the teacher presented several problems to the class in which he asked to compare the size of straws that had not been made by the students. For instance, in a problem presented on day 5, the students were asked to decide which would be longer, a small of 60 or a small of 80. On every occasion, solutions consistent with the inverse order relation were treated as self-evident. When probed, students would offer reasonable justifications. The following quotes show how three students explained why the small of 40 would have to be bigger than the small of 60:

> Lupita: Because they [smalls of 40] take more space? [“Porque ocupan más lugar?”]
>
> Vicky: It [the small of 40] covers more on the stick and it's bigger [“Cabe más en la vara y es más grande”]
>
> Ángel: Because the 60 one [the small] is smaller and the 40 one is a bit bigger [“Porque el de sesenta está más chico y el de cuarenta está un poquito más grande”]

It is noticeable that in every case, the students' explanations were quantitative in nature. By this point of the experiment, all the children produced solutions to these kinds of comparisons, which reflected their understanding of both parts of the specification we have given for the inverse order relationship.

As previously mentioned, the students were individually interviewed after the final session of the design experiment (i.e., the 13th session). By then, the use of canonical fraction inscriptions had been established in the classroom. The interviews included two questions directly related to the inverse order relationship. In one of them, students were presented with cards containing the inscriptions 1/11 and 1/7 and asked to explain which represented the bigger amount.

All of the students produced solutions consistent with the inverse order relationship. Two of them made the comparison directly. The rest had to be supported to interpret the inscriptions as representing the sizes of smalls. All of the children gave reasonable explanations of their solutions. The following example comes from the interview of one of the students who showed the least sophisticated reasoning during the design experiment:

Teacher: Which do your think would be bigger?

Ángel: [Touching the card showing 1/7] This one.

Teacher: Why?

Ángel: Because it fits seven times [on Tikhe] and the other fits 11 times. It's [1/11] very small [gesturing with his fingers] and this one [1/7] is a little bit bigger [gesturing with his fingers].

Limitations in students' understandings

As the instructional sessions continued, it became apparent that the way in which most of the students conceived the relation between the size of a subunit and the number of iterations was unidirectional. Thereby, by day 5, the students would imagine the size of, say, a small of ten, as an entity of such a length that ten iterations of it would be necessary to cover the length of the unit. This image allowed them to compare any two subunits in ways consistent with the inverse order relationship. However, this image, for most of the students, did not readily imply that the inverse needed to be true; that is, that ten iterations of a small of ten would necessarily have to render a length as long as the unit.

The research team had not anticipated that it would be rather difficult for the students to recognize this reciprocal relation. In the following week, the main instructional goal became to help students understand it. Our data shows that by day 10, the class treated this relation as self-evident. Students drew on it to assess and compare measures that entailed the iteration of subunits. For instance, they would draw on it to judge what would be longer, a length that corresponded to ten iterations of the small of 11 or one that corresponded to six iterations of a small of five (i.e., 10/11 vs. 6/5). The emergence of this other way of understanding the relation between a subunit and the unit established the second mathematical practice.

Conclusions

In this paper, we analyzed a classroom design experiment in which fourth grade students were supported to make sense of unit fractions, from the start, as numbers that quantify reciprocal and multiplicative relations. The results show that we were successful in helping them make sense of the inverse order relationship, a mathematical notion that is important in proportional reasoning, and in fraction learning.

Our instructional design was based on Freudenthal's formulation of the *fraction as comparer approach*. Hence, the means of support we developed characterize the entities that fractions quantify as always being separate from the reference unit. Results from the design experiment support regarding instruction based on this approach as a viable alternative to the predominant instructional strategies, in which students are led to construe unit fractions as equal size parts of food items and other objects.

By and large, the results from our study suggest that it would be possible to develop a systematic instructional approach to fractions that is inherently about reasoning about relative size. This approach would be grounded in, and contribute to, the development of students' proportional reasoning from the outset, and allow to readily engage novice fraction learners in multiplicative rather than additive forms of reasoning.

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